# Deligne's construction

Merrick Cai

September 30, 2024

## Contents



# 1 Introduction

Let $X$ be a "nice" topological space. Let $U \subset X$ be an open dense subset, with open embedding $j : U \hookrightarrow X$. If $\mathcal{F}$ is a perverse sheaf on $U$, then there should a unique way to "minimally extend" $\mathcal{F}$ to a perverse sheaf on all of $X$, preserving certain properties such that irreducibility. It turns out this is true, and the functor is called the intermediate extension, denoted by $j_{!*}$. More generally, let $\iota : V \hookrightarrow X$ be a locally closed embedding. Then to any perverse sheaf $\mathcal{F}$ on $V$, there is a functor $h_{!*}$ extending it to a perverse sheaf on $X$, supported on $\overline{V}$, and satisfying certain important properties (for example, if $\mathcal{F}$ was irreducible, then so is $h_{!*}\mathcal{F}$).

Deligne gave an explicit construction of the intermediate extension, which is very roughly stated as:

**Theorem 1.1 (Deligne construction):** Let $\mathcal{F}$ be a perverse sheaf on $U$. Then

$$j_{!*}\mathcal{F} = \tau_{\leq -d-1} j_* \mathcal{F}.$$

(This is not exactly true, but we'll make this more precise later.)

Specifically, we want to use it in a "step by step" application of constructing $j_{!*}\mathcal{F}$ "by hand," to be explained in Corollary 1.5.

In [Wil15, Exercise 11.10], Geordie Williamson makes the following claim (I think… it's a bit hard to understand):

**Exercise 1.2:** Let $\mathcal{F}$ be a perverse sheaf on $U \subset X$. Then we can concretely realize ${}^p j_!$, $j_{!*}$, and ${}^p j_*$ as

(a) ${}^p j_! \mathcal{F} = \tau_{\le -d-2} j_* \mathcal{F}$.
(b) $j_{!*} \mathcal{F} = \tau_{\le -d-1} j_* \mathcal{F}$.
(c) ${}^p j_* \mathcal{F} = \tau_{\le -d} j_* \mathcal{F}$.

Once again, we'll make this more precise in §1.2.

I couldn't really find a reference for this, but everyone I asked said they'd never seen it before, so I'm inclined to think that it's at least not easy to track down in the literature. Anlong Chua and I spent a while trying to figure it out, so I figured I'd write it up so I don't have to go through the pains of reproving it.

## 1.1 The condition

The setting we will need is a bit more nuanced. Recall that a **perverse sheaf** on $X$ is a complex of sheaves, constructible with respect to *some* stratification of $X$, satisfying perverse sheaf conditions outlined in Lemma 1.10. If we **specify the stratification**, however, then we get a subcategory consisting of complexes which are constructible specifically with respect to the given stratification. Therefore there are two cases: either we specify the stratification, or we don't.

### 1.1.1 Specifying the stratification

Let

$$X = \bigsqcup X_\lambda$$

be a stratification of $X$, such that the star-pushforward along the inclusion of any stratum sends a local system of finite type to a constructible complex (with respect to the stratification). This is what [Ach21, Definition 2.3.20] calls a "good stratification;" it satisfies the important property that for any inclusion $h : Y \hookrightarrow X$ where $Y$ is a union of strata, then $h_*, h_!, h^*, h^!$ all send constructible complexes (with respect to the stratification) to constructible complexes (with respect to the stratification), see [Ach21, Lemma 2.3.22].

Our **specific condition** that we need to impose, is that there exists $d$ such that

$$U = \bigsqcup_{\dim X_\lambda > d} X_\lambda,$$

$$Z = \bigsqcup_{\dim X_\lambda = d} X_\lambda,$$

and **there exist no strata of dimension** $< d$.

**Remark 1.3:** One important consequence of this is that $Z$ is genuinely the disjoint union of the strata which comprise it. The reason is that each stratum inside of it is closed (as they are minimal dimension, hence their closure must be a union of itself and lower-dimensional strata, of which there are none), so $Z$ is set-wise the disjoint union of closed subsets - but then viewing them as complements to the other strata in $Z$, they must also be open in $Z$, hence are connected components of $Z$.

### 1.1.2 Not specifying the stratification

The difference is that now a perverse sheaf just needs to be constructible with respect to *some* stratification. In this case, let $\mathcal{F}$ be a perverse sheaf on $U$; it is a constructible complex of sheaves on $U$. Let $j : U \hookrightarrow X$ be the open embedding. Then $j_*$ sends constructible complexes to constructible complexes, so $j_*\mathcal{F}$ is a constructible complex of sheaves on $X$. Suppose it's constructible with respect to some (good) stratification $\{X_\lambda\}$, such that $U$ is also a union of strata (we can always refine the stratification until this works). The **specific condition** that we need to impose here is that there exists $d$ such that

$$U = \bigsqcup_{\dim X_\lambda > d} X_\lambda,$$

$$Z = \bigsqcup_{\dim X_\lambda = d} X_\lambda,$$

and **there exist no strata of dimension** $< d$.

### 1.1.3 Conclusion

In summary, whether the stratification is given or not, **we really need the complement** $Z$ **of** $U$ **to be a union of strata which are all the same dimension** $d$. For now, feel free to imagine that

$$Z = \bigsqcup_{\dim X_\lambda \leq d} X_\lambda = X - U,$$

and I'll point out where it becomes necessary for this hypothesis.

## 1.2 A more precise formulation

Let us now precisely describe Deligne's construction.

**Theorem 1.4 (Deligne's construction, reformulated):** Let $\mathcal{F}$ be a perverse sheaf on $U \subset X$. If the stratification is given, then we assume the condition in §1.1.1. If the stratification is not given, then we assume the condition in §1.1.2. Then

$$j_{!*}\mathcal{F} = \tau_{\leq -d-1} j_* \mathcal{F}.$$

As a direct corollary, we can drop the hypothesis coming from §1.1 by iterating "step by step" over the dimensions. However, first we reduce the problem a bit.

Let $X$ be a "nice" topological space, and $U \subset X$ be some dense open subset of $X$ with open embedding $j : U \hookrightarrow X$. Let $\mathcal{F}$ be a perverse sheaf on $U$. Now we aim to compute $j_{!*}\mathcal{F}$; we know that $j^* j_{!*}\mathcal{F} \simeq \mathcal{F}$, and in particular for any open $V \subset U$ with inclusion $\widetilde{j} : V \hookrightarrow X$, we have

$$\widetilde{j}_{!*}\left(\mathcal{F}|_V\right) \simeq j_{!*}\mathcal{F}.$$

Now since $\mathcal{F}$ is constructible, we know that there exists some open dense (smooth) $V \subset U$ for which $\mathcal{F}|_V$ is a local system, shifted by $\dim V$. Hence to compute $j_{!*}\mathcal{F}$ it suffices to assume that $\mathcal{F}$ is a shifted local system. (Note that if we're given a (good) stratification, as in the situation of §1.1.1, then it's even easier: just choose the open stratum $U$, where $\mathcal{F}|_U$ is a local system shifted by $\dim U$.)

**Corollary 1.5 (constructing $j_{!*}$ step by step):** Let $j : U \hookrightarrow X$ be a (dense) open embedding, and $\mathcal{F} = \mathcal{L}[\dim U]$ is a perverse sheaf on $U$ which is specifically the shift of a local system $\mathcal{L}$ on $U$. Then $j_*\mathcal{F}$ is constructible with respect to a stratification

$$U \sqcup \bigsqcup Z_\lambda$$

where $\bigsqcup Z_\lambda = X - U$. Define $U_i$ for $i = 0, 1, \ldots, \dim U$ to be

$$U_i \coloneqq U \sqcup \bigsqcup_{\dim Z_\lambda \geq i} Z_\lambda,$$

so that we have an increasing union

$$U = U_{\dim U} \xhookrightarrow{j_{\dim U}} U_{\dim U - 1} \xhookrightarrow{j_{\dim U-1}} \cdots \xhookrightarrow{j_1} U_0 = X,$$

and

$$U_i - U_{i+1} = \bigsqcup_{\dim Z_\lambda = i} Z_\lambda.$$

Then

$$j_{!*}(\mathcal{F}) = \left(\tau_{\leq -1} \circ j_{1,*}\right) \circ \left(\tau_{\leq -2} \circ j_{2,*}\right) \circ \cdots \circ \left(\tau_{\leq -\dim U - 1} \circ j_{\dim U - 1,*}\right) \circ \left(\tau_{\leq -\dim U} \circ j_{\dim U,*}\right)(\mathcal{F}).$$

*Proof.* Since

$$U_i - U_{i+1} = \bigsqcup_{\dim Z_\lambda = i} Z_\lambda.$$

we are in the setting of the Deligne construction, and can apply the Deligne construction (1.4) iteratively to the chain of inclusions

$$U = U_{\dim U} \xhookrightarrow{j_{\dim U}} U_{\dim U - 1} \xhookrightarrow{j_{\dim U-1}} \cdots \xhookrightarrow{j_1} U_0 = X.$$

□

**Remark 1.6:** Amusingly, this covers the case when there is some integer $t$ for which *no* strata are of dimension $t$, because we still *treat* the empty complement as having dimension $t$.

Let us reformulate the exercise.

**Exercise 1.7:** Let $\mathcal{F}$ be a perverse sheaf on $U \subset X$. If the stratification is given, then we assume the condition in §1.1.1. If the stratification is not given, then we assume the condition in §1.1.2.

Then we can concretely realize ${}^p j_!$, $j_{!*}$, and ${}^p j_*$ as

(a) ${}^p j_! \mathcal{F} = \tau_{\leq -d-2} j_* \mathcal{F}$.
(b) $j_{!*}\mathcal{F} = \tau_{\leq -d-1} j_* \mathcal{F}$.
(c) ${}^p j_* \mathcal{F} = \tau_{\leq -d} j_* \mathcal{F}$.

This allows us to construct them "step by step," exactly as in Corollary 1.5.

Of course, part (b) is Deligne's construction (1.4); see for example [HTT07, Proposition 8.2.11].

### 1.3 Facts we will need

**Proposition 1.8 ($t$-exactness):** Let $i : Z \hookrightarrow X$ be a closed embedding.

- $i_! = i_*$ is $t$-exact for both the standard and perverse $t$-structures.
- $i^*$ is $t$-exact for the standard $t$-structure and right $t$-exact for the perverse $t$-structure.
- $i^!$ is left $t$-exact for both the standard and perverse $t$-structures.

Let $j : U \hookrightarrow X$ be an open embedding.

- $j_!$ is $t$-exact for the standard $t$-structure and right $t$-exact for the perverse $t$-structure.
- $j_*$ is left $t$-exact for both the standard and perverse $t$-structures.
- $j^* = j^!$ is $t$-exact for both the standard and perverse $t$-structures.

**Remark 1.9:** This fails when $f$ is not a locally closed embedding. For example in general $f_!$ is only left $t$-exact, but if $f$ is a locally closed embedding, then $f_!$ is indeed $t$-exact.

**Lemma 1.10 ([HTT07, Proposition 8.1.22]):**
(a) $F \in {}^pD^{\leq 0}(X)$ iff $H^j(\iota^*_{X_\lambda} F) = 0$ for all strata $X_\lambda$ and all $j > -\dim X_\lambda$.
(b) $F \in {}^pD^{\geq 0}(X)$ iff $H^j(\iota^!_{X_\lambda} F) = 0$ for all strata $X_\lambda$ and all $j < -\dim X_\lambda$.

# 2 ${}^p j_! \mathcal{F}$

Exercise 1.7(a) asks us to prove that (under suitable conditions)

$$ {}^p j_! \mathcal{F} = \tau_{-d-2} j_* \mathcal{F}. $$

Unfortunately, this is just false.

**Example 2.1:** Consider $X$ the disk, with $Z = \{0\}$ and $U = X - \{0\}$. Then let $\mathcal{F}$ be any local system shifted by $[1]$. It's known that $j_*\mathcal{F}$ and $j_!\mathcal{F}$ are perverse sheaves in this scenario; we can also easily calculate its table of stalks.

The main point is that $d = 0$ here, and if $j_*\mathcal{F}$ is perverse, then it still must be concentrated in degrees $[-1, 0]$ (in the standard $t$-structure). Then $\tau_{\leq -d-2} = \tau_{\leq -2}$ always kills it.

So if indeed ${}^p j_! \mathcal{F} = \tau_{\leq -d-2} j_* \mathcal{F}$, then ${}^p j_! \mathcal{F} = \tau_{\leq -2} j_* \mathcal{F} = 0$. But on the other hand $j_{!*}\mathcal{F}$ is *defined* to be the image (in the abelian category of perverse sheaves) of ${}^p j_! \mathcal{F} \longrightarrow {}^p j_* \mathcal{F}$, and $j^* j_{!*}\mathcal{F} \simeq \mathcal{F} \neq 0$, hence $j_{!*}\mathcal{F} \neq 0$. But clearly we can't have something nonzero arising as the image of the zero object.

**Remark 2.2:** It's possible there's a correct formulation involving a truncation of $j_!$, but I haven't worked out the details (and I'm a bit dubious). It can't be a truncation of $j_!$, since $j_!$ is right $t$-exact in the perverse $t$-structure, hence ${}^p j_! \neq j_!$ in general, but $j_!$ is $t$-exact in the standard $t$-structure, so it'll commute with any truncation functor. On the other hand for truncating $j_*$, it has to be $\tau_{\leq ?}$ since $j_*$ is left $t$-exact in both $t$-structures, but it can't be $-d-1$ or $-d$ since those correspond to $j_{!*}$ and ${}^p j_*$. But less than that and we'll run into issues where if $\operatorname{codim}_Z(X) > 1$, we might just delete the entire sheaf.

# 3 ${}^p j_* \mathcal{F}$

Let's prove that

$$ {}^p j_* \mathcal{F} = \tau_{\leq -d} j_* \mathcal{F}. $$

In the proofs, **we will actually assume that $Z$ is the union of strata of dimension $\leq d$**, rather than being purely of strata of dimension $d$. The reason is that we can; we will specifically point out where we need the stronger hypothesis (that $Z$ is the union of strata all of which are dimension $d$) and highlight it in blue; only in these places is it necessary to assume.

**Claim 3.1:** $\tau_{\leq -d} j_* \mathcal{F}$ is a perverse sheaf.

*Proof.* We need to check that $\tau_{\leq -d} j_* \mathcal{F} \in {}^p D^{\leq 0} \cap {}^p D^{\geq 0}$.

- First we check ${}^p D^{\leq 0}$; we will use the condition from Lemma 1.10(a). Let's consider when $X_\lambda \subset U$ and $X_\lambda \subset Z$.
  - If $X_\lambda \subset U$, then we have

    $$\iota^*_{X\lambda} \tau_{\leq -d} j_* \mathcal{F} = \iota^*_{X\lambda} j^* \tau_{\leq -d} j_* \mathcal{F} = \iota^*_{X\lambda} \tau_{\leq -d} j^* j_* \mathcal{F} = \iota^*_{X_\lambda} \tau_{\leq -d} \mathcal{F},$$

    since $j^*$ is $t$-exact (since it's inclusion of open subset). Then since $\mathcal{F}$ is perverse on $U$, all of whose strata are dim $> d$, it's clear that $\mathcal{F}$ is concentrated in degrees $< -d$, hence $\tau_{\leq -d} \mathcal{F} = \mathcal{F}$. Then the condition that $\iota^*_{X_\lambda} \mathcal{F}$ has no cohomologies of degrees $> -\dim X_\lambda$ is implied by $\mathcal{F}$ being perverse on $U$.
  - If $X_\lambda \subset Z$, then we need to check that

    $$\tau_{> -\dim X_\lambda} \iota^*_{X_\lambda} \tau_{\leq -d} j_* \mathcal{F} = 0.$$

    But $\iota^*$ is $t$-exact so this is

    $$\tau_{> -\dim X_\lambda} \tau_{\leq -d} \iota^*_{X_\lambda} j_* \mathcal{F},$$

    and since $\dim X_\lambda \leq d$, the truncation functors cross and this is zero.
- Now let's check ${}^p D^{\geq 0}$; we will use the condition from Lemma 1.10(b). Let's consider when $X_\lambda \subset U$ and $X_\lambda \subset Z$.
  - Suppose $X_\lambda \subset U$. Following the above, we have

    $$\iota^!_{X_\lambda} \tau_{\leq -d} j_* \mathcal{F} = \iota^!_{X_\lambda} j^! \tau_{\leq -d} j_* \mathcal{F} = \iota^!_{X_\lambda} \tau_{\leq -d} j^* j_* \mathcal{F} = \iota^!_{X_\lambda} \tau_{\leq -d} \mathcal{F},$$

    since $j^! = j^*$. But once again $\tau_{\leq -d} \mathcal{F} = \mathcal{F}$ so the condition is implied by $\mathcal{F}$ being perverse on $U$.
  - Now suppose $X_\lambda \subset Z$. We need to check that

    $$\tau_{< -\dim X_\lambda} \iota^!_{X_\lambda} \tau_{\leq -d} j_* \mathcal{F} = 0.$$

    Now we have an exact triangle

    $$\tau_{\leq -d} j_* \mathcal{F} \to j_* \mathcal{F} \to \tau_{> -d} j_* \mathcal{F} \xrightarrow{+1}.$$

    Applying the exact functor $\iota^!_{X_\lambda}$, we get

    $$\iota^!_{X_\lambda} \tau_{\leq -d} j_* \mathcal{F} \to \iota^!_{X_\lambda} j_* \mathcal{F} \to \iota^!_{X_\lambda} \tau_{> -d} j_* \mathcal{F} \xrightarrow{+1}.$$

    Since $\iota^!_{X_\lambda} j_* = 0$, the middle term is zero. Thus we get an isomorphism

    $$\iota^!_{X_\lambda} \tau_{\leq -d} j_* \mathcal{F} \simeq \iota^!_{X_\lambda} \tau_{> -d} j_* \mathcal{F}[-1] \implies \tau_{< -\dim X_\lambda} \iota^!_{X_\lambda} \tau_{\leq -d} j_* \mathcal{F} \simeq \tau_{< -\dim X_\lambda} \left( \iota^!_{X_\lambda} \tau_{> -d} j_* \mathcal{F}[-1] \right).$$

    Note that the term on the right is rewritten (after "commuting" the truncation with the shift) as

    $$\left( \tau_{\leq -\dim X_\lambda} \iota^!_{X_\lambda} \tau_{> -d} j_* \mathcal{F} \right) [-1].$$

    But $\tau_{> -d} j_* \mathcal{F}$ is concentrated in degrees $> -d$, and $\iota^!_{X_\lambda}$ is left $t$-exact, hence $\iota^!_{X_\lambda} \tau_{> -d} j_* \mathcal{F}$ is still concentrated in degrees $> -d$; but then $\tau_{\leq -\dim X_\lambda}$ kills it if $\dim X_\lambda \geq d$, i.e., $\dim X_\lambda = d$ for all $\lambda$. This is exactly where we need the hypothesis that $Z$ is a union of strata which are all of the same dimension.

□

**Remark 3.2:** Note that truncation functors are not triangulated functors, i.e., do not send exact triangles to exact triangles.

**Remark 3.3:** In general, I would imagine that the condition

$$\tau_{<-\dim X_\lambda} \iota^!_{X_\lambda} \tau_{\leq -d} j_* \mathcal{F} = 0$$

for all $X_\lambda$ strata in $Z$ is false without the hypothesis that all $X_\lambda$ satisfy $\dim X_\lambda = \dim Z$, however I don't have a counterexample here.

Now we return to proving that ${}^p j_* \mathcal{F} \simeq \tau_{\leq -d} j_* \mathcal{F}$.

*Proof of Exercise 1.7(b).* Now that we know it's a perverse sheaf, we have the following exact triangle via (standard) truncation functors:

$$\underbrace{\tau_{\leq -d} j_* \mathcal{F}}_{\star} \to j_* \mathcal{F} \to \underbrace{\tau_{> -d} j_* \mathcal{F}}_{\square} \xrightarrow{+1} .$$

Then by the long exact sequence of perverse cohomology functors and using the claim that $\tau_{\leq -d} j_* \mathcal{F} = {}^0 H^0(\tau_{\leq -d} j_* \mathcal{F})$, we have

$$\underbrace{{}^p H^{-1}(\tau_{\leq -d} j_* \mathcal{F})}_{=0} \to \underbrace{{}^p H^{-1}(j_* \mathcal{F})}_{=0,\ j_* \mathcal{F} \in {}^p D^{\geq 0}} \to {}^p H^{-1}(\square) \to \underbrace{{}^p H^0(\star)}_{=\star} \to {}^p H^0(j_* \mathcal{F}) \to {}^p H^0(\square) \to \underbrace{{}^p H^1(\star)}_{=0} .$$

We need to verify that ${}^p H^{-1}(\square) = {}^p H^0(\square) = 0$; this will imply that $\star = {}^p H^0(j_* \mathcal{F})$, which is what we want.

**Claim 3.4:** $\square \coloneqq \tau_{> -d} j_* \mathcal{F} \in {}^p D^{\geq 1}$.

*Proof.* For this we use Lemma 1.10(b). We need to know that for all strata $X_\lambda$, that

$$\tau_{<-\dim X_\lambda + 1} \iota^!_{X_\lambda} \tau_{> -d} j_* \mathcal{F} = 0.$$

First suppose $X_\lambda \subset U$. Then

$$\dim X_\lambda > d \implies \dim X_\lambda - 1 \geq d,$$

so first we have $\tau_{> -d} j_* \mathcal{F}$ which is a complex concentrated in (standard) degrees $> -d$. Then $\iota^!_{X_\lambda}$ is left $t$-exact for the standard $t$-structure, so it's still concentrated in degrees $> -d$. Finally $\tau_{<-\dim X_\lambda + 1}$ now only takes those integers $n$ for which $n < -\dim X_\lambda + 1 \leq -d$ and $n >= d$, for which there are none; thus it's zero.

Now suppose $X_\lambda \subset Z$. Then $\dim X_\lambda \leq d$. Now applying the exact functor $\iota^!_{X_\lambda}$ to the exact triangle

$$\iota^!_{X_\lambda} \tau_{\leq -d} j_* \mathcal{F} \to \iota^!_{X_\lambda} j_* \mathcal{F} \to \iota^!_{X_\lambda} \tau_{> -d} j_* \mathcal{F} \xrightarrow{+1},$$

note that $\iota^!_{X_\lambda} j_* = 0$, hence we get

$$\iota^!_{X_\lambda} \tau_{> -d} j_* \mathcal{F} = (\iota^!_{X_\lambda} \tau_{\leq -d} j_* \mathcal{F})[1].$$

Now knowing that $\tau_{\leq -d} j_* \mathcal{F}$ is a perverse sheaf, we know that

$$\iota^!_{X_\lambda} j_* \mathcal{F} \in D^{\geq -\dim X_\lambda},$$

hence the shift by [1] yields the result. □

□

As a result, we can once again construct ${}^p j_*$ "step by step."

## 3.1 Alternate proof

We can prove it alternatively using the perverse $t$-structure exact triangle

$$ {}^p\tau_{\leq 0} j_*\mathcal{F} \to j_*\mathcal{F} \to {}^p\tau_{>0} j_*\mathcal{F} \xrightarrow{+1} . $$

*Alternative proof of Exercise 1.7(b).* Note that $j_*$ is left $t$-exact in the perverse $t$-structure, so

$$ {}^p\tau_{\leq 0} j_*\mathcal{F} = {}^p j_*\mathcal{F} $$

and $j_*\mathcal{F} \in {}^pD^{\geq 0}$. It remains to show that

$$ {}^p\tau_{>0} j_*\mathcal{F} \xrightarrow{\sim} \tau_{>-d} j_*\mathcal{F} . $$

But note that applying the $t$-exact $j^*$, the exact triangle returns

$$ \mathcal{F} \xrightarrow{\sim} \mathcal{F} \to j^{*p}\tau_{>0} j_*\mathcal{F} \xrightarrow{+1}, $$

hence $j^{*p}\tau_{>0} j_*\mathcal{F} = 0$ and ${}^p\tau_{>0} j_*\mathcal{F} = i_*\mathcal{G}$ is supported on $Z$. Now from the fact that a perverse sheaf on $Y$ is concentrated in standard cohomology degrees $-\dim Y$ to 0, we have that $\mathcal{G} \in D^{>-d}$. Since $i_*$ is $t$-exact in both $t$-structures it's also true that ${}^p\tau_{>0} j_*\mathcal{F} \in D^{>-d}$. But note that $Z$ is a disjoint union (topologically) of its strata (see Remark 1.3), so being perverse with respect to the stratification on $Z$ is just the same thing as being local systems on each component on $Z$, but shifted by $d = \dim Z$. (Here we really need the condition that $Z$ is comprised of strata all of which are the same dimension!) In other words, $\mathrm{Perv}(Z) \simeq \mathrm{Loc}(Z)[d] \subset D(Z)$: inside the derived category of sheaves on $Z$, constructible with respect to the stratification on $Z$, the abelian subcategory of perverse sheaves (with respect to the stratification on $Z$) is nothing more than the abelian subcategory of local systems on $Z$ (i.e., local systems on each component of $Z$) shifted by $d$. Therefore the two truncation functors commute for (complexes of) sheaves supported on $Z$ (and constructible with respect to the given stratification). Now we have a natural map

$$ j_*\mathcal{F} \to \tau_{>-d} j_*\mathcal{F} $$

given by the exact triangle for truncation functors from the standard $t$-structures. Now ${}^p\tau_{>0}$ is a functor (but not a triangulated functor) so we can apply it to obtain an induced map

$$ {}^p\tau_{>0} j_*\mathcal{F} \to {}^p\tau_{>0}\tau_{>-d} j_*\mathcal{F} . $$

Now since the two truncation functors do the same thing, we can kill the left truncation functor and we have an induced map

$$ {}^p\tau_{>0} j_*\mathcal{F} \to \tau_{>-d} j_*\mathcal{F} $$

of complexes supported on $Z$, which clearly induce isomorphisms on the cohomologies, which are just local systems on $Z$ (but they are obtained by degrees $> 0$ in the perverse $t$-structure, or by degrees $> -d$ in the standard $t$-structure; the point is that they are the same). So we've exhibited a map which induces isomorphisms on the cohomologies (in either $t$-structure that you want to use), hence the result. □

**Remark 3.5:** The core of this proof is that once we have complexes supported on $Z$, constructible with respect to the stratification on $Z$, then the two $t$-structures agree up to shift by $d$: they're both just local systems on $Z$, just in different (standard) degrees.

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